

ALGEBRAIC EMBEDDINGS OF \mathbb{C} INTO $\mathrm{SL}_n(\mathbb{C})$

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ABSTRACT. We prove that any two algebraic embeddings $\mathbb{C} \rightarrow \mathrm{SL}_n(\mathbb{C})$ are the same up to an algebraic automorphism of $\mathrm{SL}_n(\mathbb{C})$, provided that n is at least 3. Moreover, we prove that two algebraic embeddings $\mathbb{C} \rightarrow \mathrm{SL}_2(\mathbb{C})$ are the same up to a holomorphic automorphism of $\mathrm{SL}_2(\mathbb{C})$.

1. INTRODUCTION

There are many results concerning algebraic embeddings of some variety into the affine space \mathbb{C}^n . Let me recall two of them. Any two algebraic embeddings of a smooth affine variety X into \mathbb{C}^n are the same up to an algebraic automorphism of \mathbb{C}^n , provided that $n > 2 \dim X + 1$. This result is due to Nori, Srinivas [Sri91] and Kaliman [Kal91]. If one relaxes the condition that the automorphism of \mathbb{C}^n must be algebraic, Kaliman [Kal13] and independently, Feller and the author [FS14] proved the following improvement: Any two algebraic embeddings of a smooth affine variety X into \mathbb{C}^n are the same up to a *holomorphic* automorphism of \mathbb{C}^n , provided that $n > 2 \dim X$.

As a further development of these results, we study algebraic embeddings of \mathbb{C} into SL_n . This article can be seen as a first example to understand algebraic embeddings of a curve into an arbitrary affine algebraic variety with a large automorphism group.

In dimension zero, Arzhantsev, Flenner, Kaliman, Kutzschebauch and Zaidenberg proved that two embeddings of a finite set into any irreducible smooth affine flexible variety Z are the same up to an algebraic automorphism of Z , provided that $\dim Z > 1$ [AFK⁺13]. Our main result is based on this work.

Main Theorem (see Theorem 4 and Theorem 7). *Let $f, g: \mathbb{C} \rightarrow \mathrm{SL}_n$ be algebraic embeddings. If $n \geq 3$, then f and g are the same up to an algebraic automorphism of SL_n and if $n = 2$, then f and g are the same up to a holomorphic automorphism of SL_n .*

To the author's knowledge it is not known, whether all algebraic embeddings $\mathbb{C} \rightarrow \mathrm{SL}_2$ are the same up to an algebraic automorphism of SL_2 . Also for algebraic embeddings $\mathbb{C} \rightarrow \mathbb{C}^3$ it is an open problem, whether all these embeddings are the same up to an algebraic automorphism of \mathbb{C}^3 , see [Sha92] for potential examples that are not equivalent to linear embeddings.

In fact, in a certain sense the class of algebraic embeddings $\mathbb{C} \rightarrow \mathrm{SL}_2$ is as big as the class of algebraic embeddings $\mathbb{C} \rightarrow \mathbb{C}^3$. More precisely,

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the following holds. If $g: \mathbb{C} \rightarrow \mathbb{C}^3$, $t \mapsto (g_1(t), g_2(t), g_3(t))$ is an algebraic embedding, then one can apply a (tame) algebraic automorphism of \mathbb{C}^3 such that afterwards the polynomial g_2 divides $g_1 g_3 - 1$ and thus the following map is an algebraic embedding

$$\mathbb{C} \longrightarrow \mathrm{SL}_2, \quad t \longmapsto \begin{pmatrix} g_1(t) & (g_1(t)g_3(t) - 1)/g_2(t) \\ g_2(t) & g_3(t) \end{pmatrix}.$$

The construction of the claimed (tame) algebraic automorphism of \mathbb{C}^3 can be seen as follows. First one can apply a map of the form $(x, y, z) \mapsto (x, y + \lambda, z)$ such that afterwards the polynomial g_2 has only finitely many simple roots, say t_1, \dots, t_n . Now, it is enough to apply some (tame) algebraic automorphism of the form $(x, y, z) \mapsto (\varphi_1(x, z), y, \varphi_3(x, z))$, which sends the points $g(t_1), \dots, g(t_n)$ to the curve $\{xz = 1, y = 0\} \subseteq \mathbb{C}^3$, see [KZ99, Lemma 5.5].

The proof of the main theorem gives a method to construct the claimed automorphism. However, the proof does not produce a computer algorithm that would give such an automorphism. This is because the construction in the proof depends on certain zero sets of polynomials.

2. ALGEBRAIC AUTOMORPHISMS OF SL_n

Let us introduce first some notation. For i, j in $\{1, \dots, n\}$, we denote the ij -th entry of a matrix $X \in \mathrm{SL}_n$ by X_{ij} . The projection $\mathrm{SL}_n \rightarrow \mathbb{C}$, $X \mapsto X_{ij}$ we denote by x_{ij} .

In the first lemma, we list algebraic automorphisms of SL_n that we use constantly. The proof is straight forward.

Lemma 1. *Let $n \geq 2$ and let $i \neq j$ be integers in $\{1, \dots, n\}$. Then, for every polynomial p in the functions x_{kl} , $k \neq i$, the map*

$$\mathrm{SL}_n \longrightarrow \mathrm{SL}_n, \quad X \longmapsto E_{ij}(p(X)) \cdot X$$

is an automorphism, where $E_{ij}(a)$ denotes the elementary matrix with ij -th entry equal to a . Similarly, for every polynomial q in the functions x_{kl} , $l \neq j$, the map

$$\mathrm{SL}_n \longrightarrow \mathrm{SL}_n, \quad X \longmapsto X \cdot E_{ij}(q(X))$$

is an automorphism.

Recall that the group of tame automorphisms of \mathbb{C}^n is the subgroup of the automorphisms of \mathbb{C}^n generated by the affine linear maps and the elementary automorphisms, i.e. the automorphisms of the form

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_i + h_i(x_1, \dots, \hat{x}_i, \dots, x_n), \dots, x_n),$$

where h_i is a polynomial not depending on x_i . In the next result we list automorphisms of \mathbb{C}^n that can be lifted to automorphisms of SL_n via the projection to the first column $\pi_1: \mathrm{SL}_n \rightarrow \mathbb{C}^n$, i.e. automorphisms ψ of \mathbb{C}^n such that there exists an automorphism Ψ of SL_n (depending on ψ) that

makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{SL}_n & \xrightarrow{\Psi} & \mathrm{SL}_n \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \mathbb{C}^n & \xrightarrow{\psi} & \mathbb{C}^n . \end{array}$$

Lemma 2. *Let $n \geq 2$. Every tame automorphism of \mathbb{C}^n that preserves the origin can be lifted to some automorphism of SL_n via $\pi_1: \mathrm{SL}_n \rightarrow \mathbb{C}^n$.*

Proof. First, remark that the group of tame automorphisms of \mathbb{C}^n that preserve the origin is generated by the linear group GL_n and by the elementary automorphisms that preserve the origin. For every $A \in \mathrm{GL}_n$, the linear map $x \mapsto A \cdot x$ of \mathbb{C}^n can be lifted to the automorphism

$$\mathrm{SL}_n \longrightarrow \mathrm{SL}_n, \quad X \longmapsto A \cdot X \cdot \mathrm{diag}(1, \dots, 1, (\det A)^{-1})$$

via π_1 , where $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ denotes the $n \times n$ -diagonal matrix with entries $\lambda_1, \dots, \lambda_n$. Let ψ be an elementary automorphism of \mathbb{C}^n that preserves the origin, i.e. there exist $i \in \{1, \dots, n\}$ and polynomials $p_1, \dots, \widehat{p_i}, \dots, p_n$ in the variables $x_1, \dots, \widehat{x_i}, \dots, x_n$ such that

$$\psi(x_1, \dots, x_n) = (x_1, \dots, x_i + \sum_{j \neq i} x_j p_j(x_1, \dots, \widehat{x_i}, \dots, x_n), \dots, x_n).$$

The automorphism ψ can be lifted to some automorphism of SL_n , e.g. to the automorphism

$$\mathrm{SL}_n \longrightarrow \mathrm{SL}_n, \quad X \longmapsto \left(\prod_{j \neq i} E_{ij}(p_j(X_{11}, \dots, \widehat{X_{i1}}, \dots, X_{n1})) \right) \cdot X,$$

cf. also Lemma 1. This finishes the proof. \square

3. A GENERIC PROJECTION RESULT

Let V be an algebraic variety. We say that a statement is true for *generic* $v \in V$ if there exists a Zariski dense open subset $U \subseteq V$ such that the statement is true for all $v \in U$.

Lemma 3. *Let $n \geq 3$. If $f: \mathbb{C} \rightarrow \mathrm{SL}_n$ is an algebraic embedding such that the matrices $f(0) - f(1)$ and $f'(0)$ have maximal rank, then, for generic $A \in \mathrm{M}_{n,n-1}$ the map*

$$\mathbb{C} \xrightarrow{f} \mathrm{SL}_n \xrightarrow{\pi_A} \mathrm{M}_{n,n-1}$$

is an algebraic embedding, where $\mathrm{M}_{n,n-1}$ denotes the space of $n \times (n-1)$ -matrices and π_A is given by $X \mapsto X \cdot A$.

Proof. Let $\Delta \subseteq \mathbb{C}^2$ be the diagonal. Consider the following (Zariski) locally closed subsets of $\mathbb{C}^2 \setminus \Delta$:

$$C_i = \{ (t, r) \in \mathbb{C}^2 \setminus \Delta \mid \mathrm{rank}(f(t) - f(r)) = i \}.$$

Consider for every $A \in \mathrm{M}_{n,n-1}$ the composition

$$(*) \quad C_i \longrightarrow \mathbb{C}^2 \setminus \Delta \xrightarrow{(t,r) \mapsto f(t) - f(r)} \mathrm{M}_{n,n} \xrightarrow{\pi_A} \mathrm{M}_{n,n-1}.$$

This map is never zero for generic $A \in \mathrm{M}_{n,n-1}$; indeed:

- If $1 < i \leq n$, then $(*)$ is never zero provided that $A \in M_{n,n-1}$ has maximal rank.
- If $i = 1$, then $\dim C_1 \leq 1$, since $\dim C_n = 2$ (note that $f(0) - f(1)$ has maximal rank). For $(t, r) \in C_1$, let $Z_{(t,r)} = \ker(f(t) - f(r))$. Since $\dim C_1 \leq 1 < n - 1$, a generic $(n - 1)$ -dimensional subspace of \mathbb{C}^n is different from $Z_{(t,r)}$ for all $(t, r) \in C_1$. Thus, for generic A the composition $(*)$ is never zero.

Clearly, $C_0 = \emptyset$. Hence, we proved that the composition $\pi_A \circ f$ is injective for generic $A \in M_{n,n-1}$. Clearly, $\pi_A \circ f$ is proper for generic $A \in M_{n,n-1}$.

For the immersivity, we have to show for generic $A \in M_{n,n-1}$ that

$$(1) \quad f'(t) \cdot A \neq 0$$

for all $t \in \mathbb{C}$. Since $\text{rank } f'(0) = n$, the set $U = \{t \in \mathbb{C} \mid \text{rank } f'(t) = n\}$ is Zariski dense and open in \mathbb{C} . Thus (1) is satisfied for all $A \neq 0$ and for all $t \in U$. Since f is immersive, we have $f'(t) \neq 0$ for all $t \in \mathbb{C}$. This implies that for generic A we have $f'(t) \cdot A \neq 0$ for all $t \in \mathbb{C}$. \square

4. ALGEBRAIC EMBEDDINGS OF \mathbb{C} INTO SL_n FOR $n \geq 3$

Theorem 4. *For $n \geq 3$, any two algebraic embeddings of \mathbb{C} into SL_n are the same up to an algebraic automorphism of SL_n .*

Lemma 5. *Let $n \geq 2$. Assume that $f: \mathbb{C} \rightarrow \text{SL}_n$ is an algebraic embedding such that*

$$\mathbb{C} \xrightarrow{f} \text{SL}_n \xrightarrow{\pi_{n-1}} M_{n,n-1}$$

is an algebraic embedding, where π_{n-1} denotes the projection to the first $n - 1$ columns. Then there exists an algebraic automorphism φ of SL_n such that

$$\mathbb{C} \xrightarrow{f} \text{SL}_n \xrightarrow{\varphi} \text{SL}_n \xrightarrow{\pi_1} \mathbb{C}^n$$

is given by $t \mapsto (1, 0, \dots, 0, t)^T$.

Proof of Lemma 5. Assume that $n = 2$. Since two algebraic embeddings of \mathbb{C} into \mathbb{C}^2 are the same up to an algebraic automorphism of \mathbb{C}^2 (Abhyankar-Moh-Suzuki Theorem, see [AM75, Suz74]), one can see that there exists an algebraic automorphism of \mathbb{C}^2 that preserves the origin and changes the embedding $\pi_1 \circ f: \mathbb{C} \rightarrow \mathbb{C}^2$ to the embedding $\mathbb{C} \rightarrow \mathbb{C}^2$, $t \mapsto (1, t)$. Using the fact that every algebraic automorphism of \mathbb{C}^2 is tame (Jung's Theorem, see [Jun42]), it follows from Lemma 2 that there exists an algebraic automorphism φ of SL_2 such that $\pi_1 \circ \varphi \circ f(t) = (1, t)$.

Assume that $n \geq 3$. Let $A(t) = \pi_{n-1} \circ f(t)$. Since the kernel of $A(t)^T$ is one-dimensional for all t , the following affine variety

$$E = \{(v, t) \in \mathbb{C}^n \times \mathbb{C} \mid A(t)^T \cdot v = 0\}$$

defines the total space of a line bundle over \mathbb{C} with projection map $(v, t) \mapsto t$. Since $n \geq 3 > \dim E$, this implies that there exists a vector $v \in \mathbb{C}^n$ such that $v^T \cdot A(t)$ is non-zero for all $t \in \mathbb{C}$. Now, complete v^T to a matrix $B \in \text{SL}_n$ with last row equal to v^T . Since $n \geq 3$, there exists a permutation matrix $P \in \text{SL}_n$, with first column equal to $(0, \dots, 0, 1)^T$. After applying the automorphism $X \mapsto B \cdot X \cdot P$ of SL_n , we can assume that

- i) the map $\mathbb{C} \rightarrow \mathrm{M}_{n,n-1}$ given by $t \mapsto (f_{ij}(t))_{1 \leq i \leq n, 2 \leq j \leq n}$ is an algebraic embedding and
 ii) the vector $(f_{n2}(t), f_{n3}(t), \dots, f_{nn}(t))$ is non-zero for all $t \in \mathbb{C}$,
 where $f_{ij}(t)$ denotes the ij -th entry of the matrix $f(t)$. By ii), there exist polynomials $\tilde{p}_k \in \mathbb{C}[t]$, $2 \leq k \leq n$ such that

$$\sum_{k=2}^n f_{nk}(t) \tilde{p}_k(t) = t - f_{n1}(t).$$

By i), there exist polynomials p_k in the functions x_{ij} with $1 \leq i \leq n$, $2 \leq j \leq n$ such that $\tilde{p}_k(t) = p_k(\dots, f_{ij}(t), \dots)$. Let $\varphi: \mathrm{SL}_n \rightarrow \mathrm{SL}_n$ be the automorphism

$$X \mapsto X \cdot \begin{pmatrix} 1 & & & \\ p_2(X) & 1 & & \\ \vdots & & \ddots & \\ p_n(X) & & & 1 \end{pmatrix}.$$

Clearly, the left down corner of the matrix $\varphi \circ f(t)$ is equal to t . Now, one can construct with the aid of Lemma 2 an automorphism ψ of SL_n such that the first column of $\psi \circ \varphi \circ f(t)$ is equal to $(1, 0, \dots, 0, t)^T$. This proves the lemma. \square

Lemma 6. *Let $n \geq 2$ and let $f: \mathbb{C} \rightarrow \mathrm{SL}_n$ be an algebraic embedding such that the first column of $f(t)$ is equal to $(1, 0, \dots, 0, t)^T$. Then f is the same as*

$$\mathbb{C} \longrightarrow \mathrm{SL}_n, \quad t \longmapsto E_{n1}(t)$$

up to an algebraic automorphism of SL_n , where $E_{n1}(t)$ denotes the elementary matrix with left down corner equal to t .

Proof of Lemma 6. Let ψ be the automorphism of SL_n defined by

$$X \mapsto X \cdot f(X_{n1})^{-1} \cdot E_{n1}(X_{n1})$$

where X_{ij} denotes the ij -th entry of the matrix X . Now, one can easily check that $\psi \circ f$ is the embedding $t \mapsto E_{n1}(t)$. \square

Proof of Theorem 4. Start with an algebraic embedding $f: \mathbb{C} \rightarrow \mathrm{SL}_n$. As SL_n is flexible, for any finite set Z in SL_n there exists an automorphism of SL_n which fixes Z and has prescribed volume preserving differentials in the points of Z , see [AFK⁺13, Theorem 4.14 and Remark 4.16]. Using the fact that $\mathrm{Aut}(\mathrm{SL}_n)$ acts 2-transitively on SL_n , see e.g. [AFK⁺13, Theorem 0.1], we can assume that

$$\det(f(0) - f(1)) \neq 0 \quad \text{and} \quad \det f'(0) \neq 0.$$

Since $n \geq 3$, by Lemma 3 there exists a matrix A in $\mathrm{M}_{n,n-1}$ of maximal rank, such that $t \mapsto f(t) \cdot A$ defines an algebraic embedding of \mathbb{C} into $\mathrm{M}_{n,n-1}$. Extend A with an additional column $v \in \mathbb{C}^n$ to a $n \times n$ -matrix $(A|v)$ of determinant one. After applying the algebraic automorphism $X \rightarrow X \cdot (A|v)$ of SL_n , we can assume that the composition

$$\mathbb{C} \xrightarrow{f} \mathrm{SL}_n \xrightarrow{\pi_{n-1}} \mathrm{M}_{n,n-1}$$

is an algebraic embedding. After an algebraic coordinate change of SL_n , we can assume that the first column of $f(t)$ is equal to $(1, 0, \dots, 0, t)^T$ by

Lemma 5. Thus, up to an algebraic automorphism of SL_n , f is the same as $t \mapsto E_{n1}(t)$ by Lemma 6. This finishes the proof. \square

5. ALGEBRAIC EMBEDDINGS OF \mathbb{C} INTO SL_2

Theorem 7. *Any two algebraic embeddings $\mathbb{C} \rightarrow \mathrm{SL}_2$ are the same up to a holomorphic automorphism of SL_2 .*

Remark 8. *Since for all $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ the embeddings*

$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t \mapsto \begin{pmatrix} 1 & at + b \\ 0 & 1 \end{pmatrix}$$

are the same up to an algebraic automorphism of SL_2 , it is enough to prove Theorem 7 up to an algebraic reparametrization of the embeddings $\mathbb{C} \rightarrow \mathrm{SL}_2$.

For the proof of Theorem 7, we need the following rather technical result, which enables us to bring an arbitrary algebraic embedding $\mathbb{C} \rightarrow \mathrm{SL}_2$ in a “nice” position.

Proposition 9. *Let $f: \mathbb{C} \rightarrow \mathrm{SL}_2$ be an algebraic embedding. Then there exists a holomorphic automorphism φ of SL_2 and a constant $a \in \mathbb{C}$ such that the embedding*

$$\mathbb{C} \longrightarrow \mathrm{SL}_2, \quad t \mapsto \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix} := (\varphi \circ f)(t + a)$$

satisfies:

- (1) *for all $t \in g_{11}^{-1}(0)$ we have $g_{12}(t) = t$;*
- (2) *the map $t \mapsto (g_{11}(t), g_{21}(t))$ is a proper, bimeromorphic immersion such that the image Γ has only simple normal crossing singularities;*
- (3) *the singularities of Γ are distinguished by the first coordinate of \mathbb{C}^2 ;*
- (4) *the line $\{0\} \times \mathbb{C}$ intersects Γ transversally; in particular, Γ is smooth in every point of $\Gamma \cap \{0\} \times \mathbb{C}$;*
- (5) *the map $t \mapsto g_{11}(t)$ is polynomial.*

The proof of this proposition uses the following easy result which is a direct application of the Baire category theorem:

Lemma 10. *Let $\mathcal{H}(\mathbb{C}^n)$ be the Fréchet space of holomorphic functions on \mathbb{C}^n with the compact-open topology. If S is the countable union of closed proper subspaces of $\mathcal{H}(\mathbb{C}^n)$, then $\mathcal{H}(\mathbb{C}^n) \setminus S$ is dense in $\mathcal{H}(\mathbb{C}^n)$.*

Let $p \in \mathbb{C}^n$ and let $i \in \{1, \dots, n\}$. In our proof of Proposition 9 we use the fact that the linear functionals on $\mathcal{H}(\mathbb{C}^n)$

$$h \mapsto h(p) \quad \text{and} \quad h \mapsto D_{x_i} h(p)$$

are continuous and thus their kernels are proper closed subspaces of $\mathcal{H}(\mathbb{C}^n)$.

Additionally, we use for the proof of Proposition 9 the following, again rather technical result:

Lemma 11. *Let $f: \mathbb{C} \rightarrow \mathrm{SL}_2$ be an algebraic embedding. Then there exists an algebraic automorphism φ of SL_2 such that the embedding*

$$\mathbb{C} \longrightarrow \mathrm{SL}_2, \quad t \mapsto (\varphi \circ f)(t) = \begin{pmatrix} x(t) & y(t) \\ z(t) & w(t) \end{pmatrix}$$

satisfies:

- a) the maps $t \mapsto x(t)$ and $t \mapsto w(t)$ are non-constant polynomials;
- b) the maps $t \mapsto (x(t), z(t))$ and $t \mapsto (x(t), w(t))$ are bimeromorphic and immersive;
- c) the singularities of the image of $t \mapsto (x(t), z(t))$ lie inside $(\mathbb{C}^*)^2$;
- d) the image of $t \mapsto (x(t), z(t))$ intersects $\{0\} \times \mathbb{C}$ transversally.

Proof of Lemma 11. Clearly, we can assume that $f(0)$ is the identity matrix $E_2 \in \mathrm{SL}_2$. By [AFK⁺13, Theorem 4.14 and Remark 4.16], there exists an algebraic automorphism of SL_2 which fixes E_2 and maps the tangent vector $f'(0) \in T_{E_2} \mathrm{SL}_2 = \mathrm{Lie} \mathrm{SL}_2$ to the matrix

$$F_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{Lie} \mathrm{SL}_2 .$$

Thus we can assume that $f(0) = E_2$ and $f'(0) = F_2$. In particular, property a) is satisfied. Since $f'(t)$ is never zero and since $f'(t)$ is invertible for generic t (note that $f'(0) = F_2$ is invertible) it follows that $f'(t) \cdot v$ is non-zero for generic $v \in \mathbb{C}^2 \setminus \{(0, 0)\}$. For generic $\mu \in \mathbb{C}$, this implies that the embedding

$$t \mapsto f(t) \cdot \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

satisfies still property a) and the projection to the first column gives an immersive map. Let us fix such a μ . For generic $\lambda \in \mathbb{C}$ the embedding

$$t \mapsto f(t) \cdot \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

still satisfies property a) and the projection to the first column and the projection to the diagonal give immersive maps. Since any immersive morphism of \mathbb{C} to an irreducible affine curve is birational, we can assume that f satisfies properties a) and b). Now, for generic $a \in \mathbb{C}$ the embedding

$$(**) \quad t \mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot f(t)$$

satisfies properties a) and b) and the singularities of the image of the projection to the first column lie inside $\mathbb{C} \times \mathbb{C}^*$. Let us fix such an a . For generic $b \in \mathbb{C}$ the embedding

$$(***) \quad t \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot f(t)$$

satisfies now the properties a) to c). Let $(p(t), q(t))^T$ be the first column of the embedding (**). Then the top left entry of the embedding (***) is given by $p(t) + bq(t)$. Now, if (***) satisfies properties a) to c), then (***) satisfies property d) if and only if $p(t) + bq(t)$ has only simple roots. However, this last condition is satisfied for generic b , since $p(t) + bq(t)$ has only simple roots if and only if for all t the vector $(1, b)^T$ lies not in the kernel of the matrix

$$\begin{pmatrix} p(t) & q(t) \\ p'(t) & q'(t) \end{pmatrix}$$

and since this last matrix is invertible for generic t and never vanishes. This finishes the proof. \square

Proof of Proposition 9. Using Lemma 11 we can assume that f satisfies the properties a) to d) of Lemma 11. As a consequence of b) and c) we get that the map $t \mapsto (x(t), z(t), w(t))$ is a proper holomorphic embedding.

Let t_1, \dots, t_n be the roots of $x(t) = 0$ (which are simple according to property d)). After a reparametrization of f of the form $t \mapsto t + a$ one can assume that $w(t_i) \neq w(t_j)$ for all $i \neq j$ and $t_i \neq 0$ for all i . Let $a_i \in \mathbb{C}$ such that $e^{-a_i} = -t_i z(t_i)$ and let $b: \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial map such that $b(w(t_i)) = a_i$ and $b'(w(t)) = 0$ for all t with $x'(t) = 0$. After applying the holomorphic automorphism

$$\mathrm{SL}_2 \longrightarrow \mathrm{SL}_2, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \longmapsto \begin{pmatrix} x & e^{-b(w)}y \\ e^{b(w)}z & w \end{pmatrix}$$

we can assume that the embedding f satisfies $y(t_i) = t_i$ for all i and f still satisfies the properties a) to d).

Let ρ be the embedding $t \mapsto (x(t), y(t), z(t))$. Fix $x_0 \neq 0$ such that

I) $z(s) \neq 0$ and $x'(s) \neq 0$ for all $s \in x^{-1}(x_0)$ and

II) the maps $t \mapsto z(t)$ and $t \mapsto w(t)$ are injective on $x^{-1}(x_0)$.

Let $\{s_1, \dots, s_n\} = x^{-1}(x_0)$. With the aid of Lemma 10 one can see that there exists a holomorphic function $c: \mathbb{C}^2 \rightarrow \mathbb{C}$ that satisfies the following:

i) for all $(x, z, w) \neq (x, z, w') \in \rho(\mathbb{C})$ we have $c(x, w) \neq c(x, w')$;

ii) for all t with $x'(t) = 0$, the partial derivative $D_w c$ vanishes in $(x(t), w(t))$;

iii) for all $i = 1, \dots, n$ we have $c(0, w(t_i)) = 0$;

iv) for all integers k, q and for all 2-element sets $\{i, j\} \neq \{l, p\}$ we have

$$\begin{aligned} & [\log z(s_l) - \log z(s_p) + 2\pi i q] \cdot [c(x_0, w(s_j)) - c(x_0, w(s_i))] \\ & \neq [\log z(s_i) - \log z(s_j) + 2\pi i k] \cdot [c(x_0, w(s_p)) - c(x_0, w(s_l))] ; \end{aligned}$$

v) for all integers k and for all $i \neq j$ we have

$$\begin{aligned} & [\log z(s_i) - \log z(s_j) + 2\pi i k] \cdot [x'(s_i)c(x, w)'(s_j) - x'(s_j)c(x, w)'(s_i)] \\ & \neq [z'(s_i)x'(s_j)/z(s_i) - z'(s_j)x'(s_i)/z(s_j)] \cdot [c(x_0, w(s_j)) - c(x_0, w(s_i))] . \end{aligned}$$

Let $V \subseteq \mathbb{C}^*$ be the largest subset such that for all $x_0 \in V$ the properties I) and II) are satisfied. By property a), the complement $\mathbb{C} \setminus V$ is a closed discrete (countable) subset of \mathbb{C} . The inequalities in iv) and v) are locally holomorphic in $x_0 \in V$ after a local choice of sections s_1, \dots, s_n of the covering $x^{-1}(V) \rightarrow V$ and a local choice of the branches of the logarithms. Since V is path-connected, one can now deduce that there exists a subset $U \subseteq V$ such that $\mathbb{C} \setminus U$ is countable and for all $x_0 \in U$ the properties iv) and v) are satisfied.

According to i) and c) there exists $\lambda \in \mathbb{C}^*$ such that for all $x_1 \in \mathbb{C} \setminus U$ we have the following: If $(x_1, z, w) \neq (x_1, z', w') \in \rho(\mathbb{C})$, then $e^{\lambda c(x_1, w)}z \neq e^{\lambda c(x_1, w')}z'$. Now, let φ be the following holomorphic automorphism

$$\mathrm{SL}_2 \longrightarrow \mathrm{SL}_2, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \longmapsto \begin{pmatrix} x & e^{-\lambda c(x, w)}y \\ e^{\lambda c(x, w)}z & w \end{pmatrix}$$

and let $g = \varphi \circ f$. According to iii), g satisfies property (1) of the proposition. Property ii) implies that $t \mapsto (g_{11}(t), g_{21}(t))$ is immersive. Clearly, $t \mapsto (g_{11}(t), g_{21}(t))$ is proper and g satisfies property (5) of the proposition. By iii), it follows that g satisfies property (4) of the proposition and thus

$t \mapsto (g_{11}(t), g_{21}(t))$ is bimeromorphic. By the choice of λ , it follows for $x_1 \notin U$ that $g_{21}(t) \neq g_{21}(t')$ for all $t \neq t' \in x^{-1}(x_1)$. Since all $x_0 \in U$ satisfy iv) and v) the image of $t \mapsto (g_{11}(t), g_{21}(t))$ has only simple normal crossings, which have distinct first coordinates in \mathbb{C}^2 . This implies properties (2) and (3) of the proposition. \square

Proof of Theorem 7. Let $f: \mathbb{C} \rightarrow \mathrm{SL}_2$ be an algebraic embedding. We will prove that up to a holomorphic automorphism of SL_2 and up to an algebraic reparametrization, f is the same as the standard embedding $t \mapsto E_{12}(t)$.

After applying a holomorphic automorphism of SL_2 and performing an algebraic reparametrization we can assume that f satisfies properties (1) to (5) of Proposition 9. We denote

$$f(t) = \begin{pmatrix} x(t) & y(t) \\ z(t) & w(t) \end{pmatrix}.$$

As usual, $\pi_1: \mathrm{SL}_2 \rightarrow \mathbb{C}^2$ denotes the projection onto the first column. Let S be the (countable) closed discrete set of points $s \in \mathbb{C}^2 \setminus \{0\}$ such that $(\pi_1 \circ f)^{-1}(s) = \{s_1, s_2\}$ with $s_1 \neq s_2$, see property (2). For every s in S , it holds that $y(s_1) \neq y(s_2)$, since f is an embedding and since all simple normal crossings of the image of $\pi_1 \circ f$ lie inside $\mathbb{C}^* \times \mathbb{C}$ due to property (4). Thus, we can choose $a_s \in \mathbb{C}$ such that

$$s_1 - e^{a_s} y(s_1) = s_2 - e^{a_s} y(s_2).$$

Let $\psi_1: \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function with $\psi_1(0) = 0$ such that for all $s \in S$ we have $\psi_1(x(s_1)) = a_s$. This function exists, since $x(s_1) = x(s_2) \neq 0$ for all $s \in S$ (by property (4)), since $x((\pi_1 \circ f)^{-1}(S))$ is a closed analytic subset of \mathbb{C} (by property (5)) and since $x(s_1) \neq x(s'_1)$ for distinct s, s' of S (by property (3)). Let α_1 be the holomorphic automorphism of SL_2 defined by

$$\alpha_1 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & e^{\psi_1(x)} y \\ e^{-\psi_1(x)} z & w \end{pmatrix}.$$

By composing f with α_1 , we can assume that $s_1 - y(s_1) = s_2 - y(s_2)$ for all $s \in S$. The embedding f still satisfies the properties (1) to (5).

Let $\Gamma \subset \mathbb{C}^2$ be the image of $\pi_1 \circ f: \mathbb{C} \rightarrow \mathbb{C}^2$. By Remmert's proper mapping theorem [Rem57, Satz 23], Γ is a closed analytic subvariety of \mathbb{C}^2 . Now, using that $\pi_1 \circ f$ is immersive and Γ has only simple normal crossings, we get a holomorphic factorization

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\pi_1 \circ f} & \Gamma \\ & \searrow t \mapsto t - y(t) & \vdots e \\ & & \mathbb{C} \end{array}$$

Using properties (1) and (4), it follows that the map

$$\tilde{e}: \Gamma \longrightarrow \mathbb{C}, \quad (x, z) \longmapsto \frac{e(x, z)}{x}$$

is holomorphic. Using Cartan's Theorem B [Car53, Théorème B], we can extend \tilde{e} to a holomorphic map $\psi_2: \mathbb{C}^2 \rightarrow \mathbb{C}$. Let α_2 be the holomorphic

automorphism of SL_2 defined by

$$\alpha_2 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y + x\psi_2(x, z) \\ z & w + z\psi_2(x, z) \end{pmatrix}.$$

After applying the automorphism α_2 to f we can assume that $y(t) = t$. This implies that $x(0)w(0) = 1$. Let p, q be the holomorphic functions such that $p(t)t = x(t) - x(0)$ and $q(t)t = w(t) - w(0)$. After applying the holomorphic automorphism

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} w(0) & 0 \\ 0 & x(0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q(y) & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p(y) & 1 \end{pmatrix}$$

we can additionally assume that $w(t) = x(t) = 1$, which implies $z(t) = 0$. The statement follows now from Remark 8. \square

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